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AN ALTERNATE SUBSET APPROACH TO THE RANKING AND SELECTION PROBLEM*

by

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1. Introduction.

We are given $k > 2$ populations π_i $(i = 1,2,\ldots,k)$, which serve as sources of observations which are independent within (as well as between) populations. The random variable X_i associated with π_i has distribution function (d.f.) $F = F(x|\theta_i)$. The functional form of F is known except for the value of the unknown scalar parameters $\theta_1,\theta_2,\ldots,\theta_k$. It is assumed for each i that $\theta_i \in \Theta$, an open interval of the real line. The (unknown) ordered θ-values are denoted by

$$(1.1) \qquad \theta_{[1]} \le \theta_{[2]} \le \ldots \le \theta_{[k]}$$

and it is assumed that there is no a priori information available about the correct pairing of the populations with the $\theta_{[i]}$-values. Let t be any fixed positive integer such that $t < k$. If $\theta_{[k-t+1]} > \theta_{[k-t]}$, then the t best populations are defined to be those with the largest θ-values; otherwise it is any set with t largest θ-values. Random samples of given common size n are taken from each of the k populations. Let P^* with $1 > P^* > \binom{k-t}{s-t}\Big/\binom{k}{s}$ and $d^* \ge 0$ denote preassigned constants and let $d = d(\theta_{[2]},\theta_{[1]})$ denote a suitable measure of distance between the populations with parameters $\theta_{[1]} \le \theta_{[2]}$.

The problem is to find the smallest subset we can select (say, of size $s \ge t$) and the associated procedure (say, R) such that the subset will contain the t best populations with probability (PCS) at least P^* whenever $d(\theta_{[k-t+1]},\theta_{[k-t]}) \ge d^*$.

We solve the problem in some generality and later find specific results and compute tables for specific families of distributions $F(x|\theta)$. The measure of distance d will have some general properties but its specific form will depend on the given family $F(x|\theta)$.

For the case of normal distributions with a common known variance (say, $\sigma^2 = 1$), this problem can be regarded as a generalization of the inverse of the problem considered by Bechhofer [1]. In the latter, t is given and the common sample size n is determined for selecting the t best. In this paper t and n are both given and we determine the required number s of populations (with $t \le s \le k$) to be selected so that we can assert with probability at least P^* that they contain the t best populations if the true configuration $\vec{\theta}$ is such that $d(\theta_{[k-t+1]}, \theta_{[k-t]}) \ge d^*$.

We allow d^* to be zero and our problem then resembles the "random-size subset" approach, e.g., in [2]. However, in that approach the subset size S is a chance variable while in our approach it is a constant, not determined by observations. For $d^* = 0$, unlike the approach in [2], we obtain $\binom{k-t}{s-t} \Big/ \binom{k}{s}$ as a lower bound (i.e. for $d = d^*$) of the PCS which is the value obtained by chance alone, i.e., without any observations. For $d^* > 0$ the final assertions of the approach in [2] and our "fixed-size subset" approach are different; nevertheless, for small positive values of d^*, it is instructive to compare our constant s-value with ES both for the W-configuration (all parameters equal) and for the GLF-configuration (see (3.1) below) with $d(\theta_{[k-t+1]}, \theta_{[k-t]}) = d^*$. It is shown in an illustration in section 5 below that our s-value can be less than both of these ES-values.

If d^* is sufficiently large then (for any common n) we will select $s = t$ populations and these will be the same as those selected in the "indifference zone" approach (see e.g. [1]).

The approach of our paper is very closely related to the work of Mahamunulu [3] which deals with the inverse problem of determining the common n required for given s (as well as k, t, d^*, P^* and distance measure).

2. Further Assumptions in the Formulation of the Problem.

The distance measure $d(\theta,\mu)$ is a continuous function, with lower bound $d_0 \geq 0$, defined for $\theta \geq \mu$ and such that

(i) $d(\theta,\mu) = d_0$ if and only if $\theta = \mu$,

(ii) for fixed θ, it is strictly decreasing in μ,

(iii) for fixed μ, it is strictly increasing in θ.

For the pure location parameter we take $d_0 = 0$ and then $d(\theta,\mu) \geq 0$; for the pure scale parameter we take $d_0 = 1$ and then $d(\theta,\mu) \geq 1$.

Let $X_{i1},\ldots,X_{in}$ be the random sample from Π_i and let $T_i = T(X_{i1},X_{i2},\ldots,X_{in})$ denote a statistic on which our procedure will be based $(i = 1,2,\ldots,k)$; we also write T to denote the common function for each i. The choice of T will depend on the given family $F(\cdot|\theta)$. When a sufficient statistic for θ exists (with fixed dimensionality b for all $n \geq b$) then we choose T to be some appropriate function of the sufficient statistic. Let $G(x|\theta_i)$ denote the d.f. of T_i $(i = 1,2,\ldots,k)$.

Assumption 1: The function T is chosen so that the family of distribution functions $\mathcal{G} = \{G(\cdot|\theta) : \theta \in \Theta\}$ is a stochastically increasing family of continuous d.f.'s.

We now define our procedure R in terms of the integer s to be determined later.

Procedure R: Order the T-values obtained from the k populations and select the subset that gives rise to the s largest values of T.

If we define a correct selection (CS) to mean the selection of a subset containing the t best populations (as defined above) then we can write our probability requirement as

$$P\{CS|\vec{\theta}\} \geq P^* \text{ for all } \vec{\theta} \text{ such that } d(\theta_{[k-t+1]},\theta_{[k-t]}) \geq d^*. \tag{2.1}$$

We wish to determine the integer $s = s(k,t,n,P^*,d^*)$ needed to make the procedure R explicit; we use the smallest integer s that satisfies (2.1). To see that such a value of s must exist we need only point out that the $\inf P\{CS|\vec{\theta}\} = P\{CS|\vec{\theta}\} = 1$ for $s = k$. In fact, we shall prove that the infimum of the $P\{CS|\vec{\theta}\}$ over all parameter points $\vec{\theta}$ for which $d(\theta_{[k-t+1]}, \theta_{[k-t]}) \geq d^*$ is an increasing function of s. Some of the results in [2] will be helpful in obtaining expressions for the infimum of the $P\{CS|\vec{\theta}\}$.

The numerical evaluation can either be accomplished by (lengthy) tables or by asymptotic (normal) approximations or by the use of graphs of the infimum of the $P\{CS|\vec{\theta}\}$ as a function of n, one graph for each value of s.

3. Infimum of PCS.

For given d^* and distance measure $d(\cdot,\cdot)$, we define the generalized least favorable (GLF) configuration to be any parameter point $\vec{\theta}$ such that

$$(3.1)\qquad \theta_{[1]} = \theta_{[2]} = \cdots = \theta_{[k-t]} = \theta'; \quad \theta_{[k-t+1]} = \theta_{[k-t+2]} = \cdots = \theta_{[k]} = \theta$$

where θ is an arbitrary point in Θ and $\theta' \in \Theta$ is such that $d(\theta,\theta') = d^*$. Let $Q(s,\theta)$ denote the value of the $P\{CS|\vec{\theta}\}$ at such GLF-configurations. To get an explicit expression for $P\{CS|\vec{\theta}\}$ we let $T_{(i)}$ denote the statistic from the population associated with $\theta_{[i]}$ $(i = 1,2,\ldots,k)$. It is easily seen that a CS occurs if and only if

$$(3.2)\qquad \min_{k-t+1\leq j\leq k} T_{(j)} > \text{at least } k-s \text{ of } (T_{(1)},T_{(2)},\ldots,T_{(k-t)})$$

or, equivalently,

$$(3.3)\qquad \min_{k-t+1\leq j\leq k} T_{(j)} > (s-t+1)^{st} \text{ largest of } (T_{(1)},T_{(2)},\ldots,T_{(k-t)}).$$

Using (3.3) it follows that for $s < k$

$$(3.4)\qquad Q(s,\theta) = \binom{k-t}{k-s}\int_{-\infty}^{\infty}[1-G(x|\theta)]^t[1-G(x|\theta')]^{s-t}dG^{k-s}(x|\theta')$$

and for $s = k$ we obtain $Q(k,\theta) \equiv 1$ for all $\theta \in \Theta$.

Using assumption 1 above it follows from the theorem of Section 4 in [2] that

$$(3.5) \qquad \inf_{\vec{\theta}\in\Omega(d^*)} P\{CS|\vec{\theta}\} = \inf_{\theta\in\Theta} Q(s,\theta)$$

where $\Omega(d^*)$ is the set of points $\vec{\theta}$ such that $d(\theta_{[k-t+1]},\theta_{[k-t]}) \geq d^*$. Hence to find the infimum on the left side of (3.5) we need only find the infimum of $Q(s,\theta)$ given in (3.4) as a function of θ.

To carry the solution further we have to know the particular class of distributions to which $G(\cdot|\theta)$ belongs. We consider two important cases in some detail, using a common notation for both cases.

Case 1: θ is a location parameter for G.

In this location problem we write $G(x|\theta) = G(x-\theta)$ and define the distance measure as $d_L(a,b) = a-b \geq 0$. For $t \geq s \geq k-1$ equation (3.4) reduces to

$$(3.6) \qquad Q_L(s) = \binom{k-t}{k-s} \int_{-\infty}^{\infty} [1-G(x-d^*)]^t [1-G(x)]^{s-t} dG^{k-s}(x).$$

It is easily verified that for $s = t$ this reduces to the well-known form

$$(3.7) \qquad Q_L(t) = t\int_{-\infty}^{\infty} G^{k-t}(x+d^*)[1-G(x)]^{t-1} dG(x).$$

Since (3.6) does not depend on θ it is also the $\inf Q_L(s,\theta)$ and hence, by (3.5) it is the desired infimum.

Case 2: θ is a scale parameter for G.

In this scale (S) parameter problem we write $G(x|\theta) = G(\frac{x}{\theta})$; here $G(0) = 0$. We use the distance measure $d(a,b) = a/b \geq 1$. For $t \leq s \leq k-1$ equation (3.4) reduces to

$$(3.8) \qquad Q_S(s) = \binom{k-t}{k-s} \int_{0}^{\infty} [1-G(\frac{x}{d^*})]^t [1-G(x)]^{s-t} dG^{k-s}(x).$$

Since this does not depend on θ it is also the $\inf Q_S(s,\theta)$ and the desired infimum by (3.5).

4. Existence of a Unique Solution.

To show that a unique value of s satisfying (2.1) exists for any given values of n, d^*, P^* (t, k and the distance measure are also fixed), we now prove a result about

$$(4.1) \qquad \Delta Q(s,\theta) = Q(s,\theta) - Q(s-1,\theta).$$

and about $Q(s) = \inf Q(s,\theta)$ for $\theta \in \Theta$.

Theorem 4.1. For any fixed $\theta \in \Theta$, the difference $\Delta Q(s,\theta)$ is positive. The difference $\Delta Q(s) \geq 0$ and is strictly positive if for each s the $\inf Q(s,\theta)$ is attained for some $\theta \in \Theta$.

Proof: Integrating by parts on the right side of (3.4), the first term vanishes and one of the two integrals obtained is easily seen to be $Q(s-1,\theta)$. Hence we obtain for $s < k$

$$(4.2) \qquad \Delta Q(s,\theta) = t\binom{k-t}{k-s}\int_{-\infty}^{\infty} G^{k-s}(x|\theta')[1-G(x|\theta')]^{s-t}[1-G(x|\theta)]^{t-1}dG(x|\theta)$$

and this is strictly positive for any $\theta \in \Theta$.

Let θ_i denote a sequence of θ-values such that $Q(s,\theta_i) \to Q(s)$. For any arbitrary $\epsilon > 0$ and sufficiently large n, we have

$$(4.3) \qquad Q(s)+\epsilon > Q(s,\theta_n) > Q(s-1,\theta_n) \geq Q(s-1).$$

Letting $\epsilon \to 0$, it follows that $\Delta Q(s) \geq 0$ and the strict inequality is clear by a similar argument if $Q(s)$ is attained at some $\theta \in \Theta$.

In most of the applications we note that the infimum $Q(s)$ is attained at values of $\theta \in \Theta$. It follows in all these cases that $\Delta Q(s) > 0$ and hence the s-value satisfying (2.1) will be unique. Since $Q(k) = 1$ it follows that such an s-value must always exist.

Remark 4.1: It may be of some interest to avoid the solution $s = k$ with probability one. In this case the value of $Q(s)$ for $s = k-1$ may be of special interest as it gives an upper bound on the values of P^* that can be achieved. From (3.4) we obtain

$$(4.4) \qquad Q(k-1) = \inf_{\theta\epsilon} \int_{-\infty}^{\infty} [1-G(x|\theta)]^t d\{1-[1-G(x|\theta')]^{k-t}\}.$$

For $\theta = \theta'$ it is easily seen that we get a lower bound for the right side of (4.4), namely

$$(4.5) \qquad Q(k-1) \geq 1 - \frac{t}{k} = \binom{k-t}{k-t-1} \Big/ \binom{k}{k-1} ,$$

which is the value obtained by chance alone.

Remark 4.2: Since s is restricted to be an integer, we will in general have to take an s-value such that $Q(s) > P^*$. If one insists on getting a lower bound equal to P^*, one can achieve this by adopting a randomized procedure. In fact, it suffices to randomize between two adjacent integers s and $s-1$, where $s \leq k$ is the non-randomized solution.

5. Applications to Specific Distributions.

5.1. Normal Means Problem.

We consider k normal populations Π_i $(i = 1,2,\ldots,k)$ with unknown means $\theta_1,\theta_2,\ldots,\theta_k$ and a common known variance σ^2, which we assume without loss of generality to be unity. We take T_i to be the sample mean $\overline{X}_i$ from population Π_i based on n observations and we are then in the location parameter case. From (3.6), using $\Phi(x)$ for the standard normal c.d.f., we obtain for $s < k$

$$(5.1) \qquad Q_L(s) = \binom{k-t}{k-s} \int_{-\infty}^{\infty} \Phi^t(x+\lambda)\Phi^{s-t}(x) d\{1-[1-\Phi(x)]^{k-s}\}$$

$$= \frac{(k-t)!}{(s-t)!(k-s-1)!} \int_{-\infty}^{\infty} \Phi^t(x+\lambda)\Phi^{s-t}(x)\Phi^{k-s-1}(-x)d\Phi(x) ,$$

where $\lambda = d^*\sqrt{n}$. If $\sigma \neq 1$ then we simply set $\lambda = d^*\sqrt{n}/\sigma$.

Tables of λ-values for the equation $Q_L(s) = P^*$ are given below for various values of k, t, s, and P^*. These tables yield the desired s-values for given n, d^* and P^* (k, t and the distance measure being fixed) in the following manner. Since $Q_L(s)$ increases with s and also with λ, it follows that for λ-solutions of $Q_L(s) = P^*$ with fixed P^* we shall have $\lambda(s-1) > \lambda(s)$. Hence if the computed value of $\lambda_c = d^*\sqrt{n}$ is such that

$$\lambda(s-1) > \lambda_c \geq \lambda(s) , \tag{5.2}$$

then we take the integer s as the desired solution.

Illustration: Suppose we have $k = 5$ normal populations with common $\sigma = \frac{1}{2}$ and we have a location parameter problem. We are interested in selecting a subset of size s containing the best $t = 2$ populations and we want the PCS to be at least $P^* = .95$ when $\theta_{[4]} - \theta_{[3]} \geq d^* = .1$. Suppose we take $n = 100$ observations on each of the 5 populations. Then $\lambda = d^*\sqrt{n}/\sigma = 2.0$ and by Table I we note that the required s is 4.

If the experimenter would like to attain a lower bound on the PCS of exactly $P^* = .95$ then he can perform an independent Bernoulli experiment with probabilities of .429 of success and .571 of failure. For a success he uses $s = 3$ and for a failure he uses $s = 4$; the average value of s is $3(.429) + 4(.571) = 3.57$. These probabilities (and one below) are obtained from a (forward) table of values of $Q_L(s)$ not included in this paper.

If we now set t equal to 1, keeping all other parameters the same then $s = 3$ is already sufficient since $Q_L(3) = .9758 > P^*$ and $Q_L(2) = .9277 < P^*$; the average s is $2(.536) + 3(.464) = 2.46$. The value for $Q_L(2)$ was taken from forward tables [5] which reduce to (3.6) and (8.5) when two of his values are equal (viz., $n_1 = n$). Based on the "random subset size" approach [2] for the corresponding problem, the maximum expected subset size for $t = 1$ is $kP^* = 4.75$ and the expected subset size for the corresponding GLF-configuration (3.1) with $\lambda = 2.0$ is 4.04. These two values are taken from equations (1.21) and (1.22) of [2]. Of course, the value of S depends on the observed results and will be smaller (i.e., close to one) when the largest sample mean is sufficiently greater than the other sample means. Although the final assertions are not exactly the same, this shows that at least for some small positive values of d^* our procedure can improve on existing results by reducing not only the maximum expected subset size but also the expected subset size for the corresponding GLF-configuration.

5.2. Exponential Distribution Starting-Point Problem.

We consider k exponential populations Π_i $(i = 1,2,\ldots,k)$ with unknown starting points $\theta_1,\theta_2,\ldots,\theta_k$ and a common known scale parameter $\sigma > 0$. We take T_i to be the minimum of the sample of size n from Π_i and we are then in the location parameter case. Since the distribution of T_i is again exponential with starting point θ_i and scale parameter σ/n, we obtain from (3.6) for $s < k$

$$Q_L(s) = \frac{n}{\sigma}(k-s)\binom{k-t}{k-s}e^{ntd^*/\sigma}\int_0^\infty\left(1-e^{-nx/\sigma}\right)^{k-s-1} e^{-nx(s+1)/\sigma}\,dx \tag{5.3}$$

$$= e^{ntd^*/\sigma}\binom{s}{t}/\binom{k}{t} .$$

Hence the required s-value is the smallest integer such that

$$\binom{s}{t} \geq P^*\binom{k}{t} e^{-ntd^*/\sigma} . \tag{5.4}$$

5.3. Uniform Distribution End-Point Problem.

We consider k uniform distributions Π_i $(i = 1,2,\ldots,k)$ all starting at 0 and with unknown positive endpoints $\theta_1,\theta_2,\ldots,\theta_k$. We take T_i to be the maximum of the sample of size n from π_i and we are then in the scale parameter case. The density of T_i is given by

$$(5.5) \qquad g_{\theta_i}(x) = nx^{n-1}/\theta^n \qquad \text{for } 0 \le x \le \theta$$

and by zero otherwise. Hence from (3.7) for $s < k$

$$(5.6) \qquad Q_S(s) = \binom{k-t}{k-s} \int_0^1 \left[1-\left(\frac{x}{d^*}\right)^n\right]^t (1-x^n)^{s-t}\, d\,(x^n)^{k-s}$$

$$= \frac{(k-s)}{d^{*nt}} \binom{k-t}{k-s} \int_0^1 [(d^{*n}-1)+y]^t\, y^{s-t}(1-y)^{k-s-1}\, dy$$

$$= (d^*)^{-nt} \sum_{\alpha=0}^{t} \binom{t}{\alpha}(d^{*n}-1)^{t-\alpha} \binom{s-t+\alpha}{s-t}\Big/\binom{k-t+\alpha}{k-t} .$$

The required value of s is the smallest integer which makes the last expression in (5.6) at least as large as P^*.

In several other cases the statistic T_i will have a simple distribution for certain small values of n and we can get explicit expressions for $Q(s)$. For example, if we rank exponential populations according to scale with known locations and the common n is 2 then T_i is exponentially distributed and we get a simple explicit expression for $Q_S(s)$.

6. Multivariate Unbiasedness of the Procedure R.

For convenience, in this section we drop the square brackets on the θ_i and assume that $\theta_1 \le \theta_2 \le \ldots \le \theta_k$; then π_i denotes the population associated with the ordered θ_i. Let $\alpha_1,\alpha_2,\ldots,\alpha_s$, and β be distinct integers from the set $I = (1,2,\ldots,k)$ such that for some i (determined by β)

$$(6.1) \qquad 1 \le \alpha_1 < \alpha_2 < \ldots < \alpha_{i-1} < \beta < \alpha_i < \ldots < \alpha_s \le k .$$

Let J be the complementary set $I - \{\alpha_1,\ldots,\alpha_{i-1},\beta,\alpha_i,\ldots,\alpha_s\}$. Further let $P(\alpha_1,\alpha_2,\ldots,\alpha_s)$ denote the probability of selecting the populations $\pi_{\alpha_1},\pi_{\alpha_2},\ldots,\pi_{\alpha_s}$ under the procedure R.

Lemma 6.1

Under the assumption 1 we have

$$(6.2)\qquad P(\alpha_1,\ldots,\alpha_{i-1},\alpha_i,\alpha_{i+1},\ldots,\alpha_s) \geq P(\alpha_1,\ldots,\alpha_{i-1},\beta,\alpha_{i+1},\ldots,\alpha_s) .$$

Proof: Consider the case $s < k-1$; the case $s = k-1$ requires only a remark. Let $U = \max_{\alpha\in J} T_\alpha$ and $H(u)$ be its c.d.f. From assumption 1 we have the result

$$(6.3)\qquad \beta < \alpha_i \Rightarrow \theta_\beta \leq \theta_{\alpha_i} \Rightarrow G(u|\theta_\beta) \geq G(u|\theta_{\alpha_i}) \quad \text{for all real } u,$$
$$\Rightarrow 1 - G(u|\theta_\beta) \leq 1 - G(u|\theta_{\alpha_i}) \quad \text{for all real } u$$

which we use below. Now, using (6.3) twice below,

$$(6.4)\qquad P(\alpha_1,\ldots,\alpha_{i-1},\alpha_i,\alpha_{i+1},\ldots,\alpha_s) = \Pr\{\min_{1\leq m\leq s} T_{\alpha_m} > \max(T_\beta,U)\}$$

$$= \int_{-\infty}^{\infty} H(u)G(u|\theta_\beta)d\left[1 - \prod_{m=1}^{s}\{1 - G(u|\theta_{\alpha_m})\}\right]$$

$$\geq \int_{-\infty}^{\infty} H(u)G(u|\theta_{\alpha_i})d\left[1 - \prod_{m=1}^{s}\{1 - G(u|\theta_{\alpha_m})\}\right]$$

$$= 1 - \int_{-\infty}^{\infty}\left[1 - \prod_{m=1}^{s}\{1 - G(u|\theta_{\alpha_m})\}\right]d[H(u)G(u|\theta_{\alpha_i})]$$

$$= \int_{-\infty}^{\infty} \prod_{m=1}^{s}\{1 - G(u|\theta_{\alpha_m})\}d[H(u)G(u|\theta_{\alpha_i})]$$

$$\geq \int_{-\infty}^{\infty}\left[\prod_{\substack{m=1\\ m\neq i}}^{s}\{1 - G(u|\theta_{\alpha_m})\}\right]\{1 - G(u|\theta_\beta)\}d[H(u)G(u|\theta_{\alpha_i})]$$

$$= P(\alpha_1,\ldots,\alpha_{i-1},\beta,\alpha_{i+1},\ldots,\alpha_s) .$$

When $s = k-1$, the proof is similar to the above with $H(u)$ replaced by 1 and hence need not be repeated.

Definition: Let $(\alpha_1, \alpha_2, \ldots, \alpha_s)$ and $(\beta_1, \beta_2, \ldots, \beta_s)$ be two non-identical s-tuples of I such that $\alpha_1 < \alpha_2 < \ldots < \alpha_s$; $\beta_1 < \beta_2 < \ldots < \beta_s$ and $\alpha_i \geq \beta_i$ $(i = 1, 2, \ldots, s)$. Then the subset of populations $\pi_{\beta_1}, \pi_{\beta_2}, \ldots, \pi_{\beta_s}$ is said to be inferior to the subset of populations $\pi_{\alpha_1}, \pi_{\alpha_2}, \ldots, \pi_{\alpha_s}$.

Now a repeated application of the lemma immediately gives

Theorem 6.1.

Under assumption 1

$$(6.5) \qquad P(\alpha_1, \alpha_2, \ldots, \alpha_s) \geq P(\beta_1, \beta_2, \ldots, \beta_s).$$

Thus the probability of selection any subset $\mathcal{A}$ of s populations is not less than the probability of selecting any other subset of size s, which is inferior to $\mathcal{A}$. We say that any procedure having the above property is a multivariate unbiased procedure; hence R is multivariate unbiased.

Let $q(\alpha)$ denote the probability of including the population π_α in the selected subset under the procedure R. For any pair $\alpha > \beta$ we can pair off all s-tuples containing α alone with those containing β alone. Using lemma 6.1 we easily obtain

Corollary 6.1.

Under the assumption 1, we have for $\alpha > \beta$

$$(6.6) \qquad q(\alpha) \geq q(\beta) .$$

Remark: In most of our applications we shall actually be dealing with a family $\mathcal{G}$ which is a strictly stochastically increasing family, i.e., $\theta_1 < \theta_2 \Rightarrow G(x|\theta_1) > G(x|\theta_2)$ for all real x. Then as a corollary to lemma 6.1 we can add that if $\theta_\beta < \theta_{\alpha_i}$ we obtain a strict inequality in (6.2). Then strict inequality holds in (6.5) if $\theta_{\alpha_j} > \theta_{\beta_j}$ for at least one j. Similarly, strict inequality holds in (6.6) if $\theta_\alpha > \theta_\beta$.

7. Operating Characteristics.

An appropriate criterion that can be used as a measure of efficiency of the procedure R is the expected number of the t best populations that are included in the selected subset; we denote this by EB and refer to it as the expected number of better populations. We now derive an expression for EB for the GLF-configuration (3.1). It is easy to see (using the notation of Section 6) that

$$\text{(7.1)} \qquad EB = \sum_{j=k-t+1}^{k} \Pr\{\Pi_j \text{ is included in selected subset}\}.$$

Under the GLF we note that the probability in (7.1) does not depend on j. Denoting the common value by b, we have

$$\text{(7.2)} \qquad b = \sum_{\beta=k-s}^{k-1} \Pr\{\text{exactly } \beta \text{ of the set } (T_{(1)},\ldots,T_{(j-1)},T_{(j+1)},\ldots,T_{(k)} \text{ are less than } T_{(j)}\}$$

$$= \sum_{\beta=k-s}^{k-1} \sum_{i\geq\beta+t-k} \binom{t-1}{i}\binom{k-t}{\beta-i} \int_{-\infty}^{\infty} G_\theta^i(x)[1-G_\theta(x)]^{t-1-i} G_{\theta'}^{\beta-i}(x)[1-G_{\theta'}(x)]^{k-t-\beta+i} dG_\theta(x)$$

where $G_\theta(x) = G(x|\theta)$ and $G_{\theta'}(x) = G(x|\theta')$. This can be written as

$$\text{(7.3)} \qquad b = \sum_{i=0}^{t-1} \binom{t-1}{i} \int_{-\infty}^{\infty} \left\{ \sum_{\alpha\geq k-s-i} \binom{k-t}{\alpha} G_{\theta'}^{\alpha}(x)[1-G_{\theta'}(x)]^{k-t-\alpha} \right\} G_\theta^i(x)[1-G_\theta(x)]^{t-1-i} dG_\theta(x)$$

$$= \sum_{i=0}^{t-1} \binom{t-1}{i} \int_{-\infty}^{\infty} I_{G_{\theta'}(x)}(k-s-i, s-t+i+1) G_\theta^i(x)[1-G_\theta(x)]^{t-1-i}\, dG_\theta(x) .$$

We can obtain a lower bound for EB by using the fact that $\theta' < \theta$ and hence $G_{\theta'}(x) \geq G_\theta(x)$. Putting this in (7.3) and tracing our steps back to the second member of (7.2) we obtain, after integration,

$$(7.4) \qquad EB = tb \geq t\sum_{\beta} \frac{\beta!(k-\beta-1)!}{k!} \sum_{i} \binom{t-1}{i}\binom{k-t}{\beta-i} = t\sum_{\beta} \frac{\beta!(k-\beta-1)!}{k!} \binom{k-1}{\beta} = \frac{ts}{k} .$$

This is clearly the value obtained by selecting s populations at random without looking at the observations and hence is a lower bound of EB at any parameter point $\vec{\theta}$.

If we want to consider a regret type of loss function L based essentially on the number of misclassified populations, then we define

$$(7.5) \qquad L = (t-B) + C[W - (s-t)],$$

where W is the number of the $k-t$ worst populations in the selected subset. Since $W = s-B$, then L takes the form

$$(7.6) \qquad L = (C+1)(t-B)$$

and the expected value of L is easily obtained from (7.1) and (7.3) for the GLF-configuration.

Since our procedure R is an invariant rule with respect to permutations of the lables of the populations and since our loss function L is also invariant and satisfies certain regularity conditions, it can be shown that for any point $\vec{\theta}$ the procedure R is best, i.e., it has the smallest expected loss or risk among all invariant procedures. A detailed proof of this statement is given in [4] and will not be repeated here.

8. A Dual Problem.

In a dual problem we select a subset of size s where $s \leq t$ and we assert that all of the s selected populations are included among the t best. Clearly the statement that the selected s are included among the t best populations is equivalent to the statement that the $k-s$ "non-selected" ones include the $k-t$ worst populations.

To obtain the latter we first consider the probability $Q_1(s,\theta)$ for $s \geq t$ that the selected subset of size s will contain the t worst populations for the GLF-configuration (3.1) with t and $k-t$ interchanged. The rule is now to select the populations with the s smallest T-values. A CS takes place if and only if

$$(8.1) \qquad \max_{1\leq j\leq t} T_{(j)} < \text{ at least } k-s \text{ of } (T_{(t+1)}, T_{(t+2)}, \ldots, T_{(k)}).$$

Hence it is easily seen that

$$(8.2) \qquad Q_1(s,\theta) = \binom{k-t}{k-s} \int_{-\infty}^{\infty} G_{\theta'}^{t}(x) G_{\theta}^{s-t}(x) \, d\{1-[1-G_{\theta}(x)]^{k-s}\} .$$

It now follows that the probability $Q_2(s,\theta)$ that the $k-s$ selected populations contains the $k-t$ worst populations for the GLF-configuration exactly as in (3.1) is

$$(8.3) \qquad Q_2(s,\theta) = \binom{t}{s} \int_{-\infty}^{\infty} G_{\theta'}^{k-t} \, G_{\theta}^{t-s}(x) \, d\{1-[1-G_{\theta}(x)]^{s}\} .$$

By the above argument, under the original rule R, this result (8.3) is also the PCS for selecting s populations that are all among the t best populations in the GLF-configuration (3.1).

We shall refer to the above type of goal with $s \leq t$ as goal 2 and to the original problem of this paper with $s \geq t$ as goal 1. It follows from the duality and theorem 4.1 that $Q_2(s,\theta)$ decreases with s for each $\theta \in \Theta$. We shall be interested in finding the largest integer s such that

$$(8.4) \qquad Q_2(s) = \inf_{\theta\in\Theta} Q_2(s,\theta) = \inf_{\vec{\theta}\in\Omega(d^*)} \text{PCS} \geq P^* .$$

Of course, to make the duality complete we define $P(CS|\vec{\theta})$ for $s = 0$ to be one for all $\vec{\theta}$, so that $Q_2(0) \equiv 1$. The results for Case 1 and 2, respectively, of Section 3 are

$$(8.5) \qquad Q_{2L}(s) = \binom{t}{s} \int_{-\infty}^{\infty} G^{k-t}(x+d^*)\, G^{t-s}(x)\, d\{1-[1-G(x)]^s\} ,$$

$$(8.6) \qquad Q_{2S}(s) = \binom{t}{s} \int_{0}^{\infty} G^{k-t}(xd^*)\, G^{t-s}(x)\, d\{1-[1-G(x)]^s\} .$$

We now establish a useful relationship between $Q_{2L}(s)$ and $Q_L(s)$; for this purpose it is convenient to use the notation $Q_{2L}(s;t)$ and $Q_L(s;t)$, respectively.

<u>Lemma 8.1</u>

If G is symmetric about 0 then

$$(8.7) \qquad Q_{2L}(s;t) = Q_L(k-s;k-t) .$$

<u>Proof:</u> Substituting $-x$ for x and $1-G(-x)$ for $G(x)$ in (3.6) gives

$$(8.8) \qquad Q_L(s;t) = \binom{k-t}{k-s} \int_{-\infty}^{\infty} G^{t}(x+d^*)\, G^{s-t}(x)\, d\{1-[1-G(x)]^{k-s}\} .$$

Replacing t by $k-t$ and s by $k-s$ and comparing with (8.5) gives the desired result.

It follows from lemma (8.1) for the normal distribution that if we set $Q_{2L}(s;t)$ or $Q_L(k-s;k-t)$ equal to P^*, we will get the same λ-value as a solution. Hence our tables constructed for goal 1 with $s \geq t$ can also be used for goal 2 with $s \leq t$. In particular, it follows that for a symmetric G with $s = t$ we obtain the same result by interchanging t and $k-t$ in (8.5) and/or (3.7), i.e.,

$$(8.9) \qquad t \int_{-\infty}^{\infty} G^{k-t}(x+d^*)[1-G(x)]^{t-1} dG(x) = (k-t) \int_{-\infty}^{\infty} G^{t}(x+d^*)[1-G(x)]^{k-t-1} dG(x).$$

9. Concluding Remarks.

Although we have made tables and illustrations using only the normal distribution, it is evident from assumption 1 that the results presented here have a wide applicability. It should be noted that for Cases 1 and 2 in Section 3 no further minimization is necessary to obtain $Q(s)$, but in general we do have to find the minimum of $Q(s,\theta)$ with respect to $\theta \in \Theta$.

For those cases in which $G_\theta(x)$ is not normal it may be possible to use the same expression with $G_\theta(x)$ replaced by the normal distribution as a first approximation. A correction term should then be found but these details will be published in a separate paper.

10. Acknowledgement.

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TABLE I

Values of λ for Normal Location Problem* (Goals 1 and 2)

For Selected Values of P^*, k, t and s.

Goal 1			P^*								Goal 2		
k	t	s	.500	.750	.900	.950	.975	.990	.995	.999	k	t	s
2	1	1	.0000	.9539	1.8124	2.3262	2.7718	3.2900	3.6428	4.3703	2	1	1
3	1	1	.5565	1.4338	2.2302	2.7101	3.1284	3.6173	3.9517	4.6450	3	2	2
		2	§	.3138	1.0919	1.5555	1.9565	2.4216	2.7376	3.3876			1
3	2	2	.5565	1.4338	2.2302	2.7101	3.1284	3.6173	3.9517	4.6450	3	1	1
4	1	1	.8368	1.6822	2.4516	2.9162	3.3220	3.7970	4.1224	4.7987	4	3	3
		2	.0000	.8115	1.5422	1.9797	2.3593	2.8010	3.1019	3.7228			2
		3		.0000	.7468	1.1912	1.5753	2.0204	2.3227	2.9440			1
4	2	2	1.1093	1.9037	2.6353	3.0808	3.4720	3.9323	4.2490	4.9099	4	2	2
		3	.0000	.7848	1.4937	1.9201	2.2915	2.7254	3.0221	3.6372			1
4	3	3	.8368	1.6822	2.4516	2.9162	3.3220	3.7970	4.1224	4.7988	4	1	1
5	1	1	1.0193	1.8463	2.5997	3.0552	3.4532	3.9196	4.2394	4.9048	5	4	4
		2	.2953	1.0825	1.7926	2.2184	2.5883	3.0189	3.3126	3.9193			3
		3		.4904	1.1964	1.6185	1.9844	2.4096	2.6990	3.2954			2
		4			.5280	.9619	1.3368	1.7711	2.0660	2.6720			1
5	2	2	1.3879	2.1474	2.8505	3.2805	3.6591	4.1058	4.4138	5.0584	5	3	3
		3	.5527	1.2731	1.9323	2.3321	2.6823	3.0932	3.3754	3.9628			2
		4		.4666	1.1413	1.5472	1.9009	2.3142	2.5971	3.1836			1
5	3	3	1.3879	2.1474	2.8505	3.2805	3.6591	4.1058	4.4138	5.0584	5	2	2
		4	.2806	1.0290	1.7073	2.1166	2.4740	2.8927	3.1798	3.7771			1
5	4	4	1.0193	1.8463	2.5997	3.0552	3.4532	3.9196	4.2395	4.9049	5	1	1
6	1	1	1.1526	1.9674	2.7100	3.1591	3.5517	4.0121	4.3280	4.9855	6	5	5
		2	.4927	1.2659	1.9639	2.3827	2.7466	3.1705	3.4597	4.0574			4
		3	.0000	.7650	1.4536	1.8659	2.2235	2.6394	2.9227	3.5071			3
		4		.2785	.9709	1.3846	1.7431	2.1595	2.4429	3.0266			2
		5			.3708	.7979	1.1669	1.5944	1.8845	2.4808			1

*see equations (5.1) and (8.7).

§blank spaces for λ-values indicate negative answers.

6 2 2	1.5694	2.3086	2.9948	3.4154	3.7862	4.2244	4.5270	5.1611	6 4 4
3	.8464	1.5399	2.1772	2.5650	2.9053	3.3055	3.5806	4.1544	3
4	.2570	.9480	1.5800	1.9633	2.2990	2.6930	2.9635	3.5264	2
5		.2624	.9181	1.3128	1.6569	2.0593	2.3347	2.9060	1
6 3 3	1.6657	2.3887	3.0627	3.4769	3.8429	4.2760	4.5755	5.2042	6 3 3
4	.8315	1.5128	2.1400	2.5224	2.8584	3.2545	3.5273	4.0976	2
5	.0000	.7087	1.3511	1.7391	2.0783	2.4761	2.7492	3.3179	1
6 4 4	1.5694	2.3086	2.9948	3.4154	3.7862	4.2244	4.5270	5.1612	6 2 2
5	.4634	1.1907	1.8506	2.2493	2.5980	3.0072	3.2882	3.8740	1
6 5 5	1.1526	1.9674	2.7100	3.1591	3.5518	4.0121	4.3280	4.9856	6 1 1
7 1 1	1.2568	2.0626	2.7972	3.2417	3.6303	4.0860	4.3989	5.0504	7 6 6
2	.6391	1.4030	2.0928	2.5068	2.8666	3.2859	3.5719	4.1633	5
3	.2009	.9541	1.6325	2.0388	2.3915	2.8017	3.0813	3.6581	4
4		.5517	1.2286	1.6335	1.9847	2.3929	2.6709	3.2439	3
5		.1224	.8060	1.2143	1.5680	1.9789	2.2584	2.8341	2
6			.2496	.6719	1.0367	1.4593	1.7461	2.3356	1
7 2 2	1.7022	2.4277	3.1024	3.5164	3.8818	4.3140	4.6127	5.2393	7 5 5
3	1.0430	1.7207	2.3451	2.7256	3.0599	3.4534	3.7242	4.2893	4
4	.5508	1.2187	1.8322	2.2054	2.5329	2.9178	3.1824	3.7338	3
5	.0592	.7336	1.3506	1.7250	2.0530	2.4381	2.7025	3.2527	2
6		.1146	.7579	1.1454	1.4834	1.8788	2.1495	2.7113	1
7 3 3	1.8468	2.5485	3.2051	3.6097	3.9679	4.3926	4.6867	5.3052	7 4 4
4	1.1245	1.7774	2.3820	2.7522	3.0784	3.4636	3.7295	4.2864	3
5	.5356	1.1857	1.7842	2.1494	2.4707	2.8494	3.1104	3.6561	2
6		.5032	1.1257	1.5020	1.8314	2.2181	2.4837	3.0373	1
7 4 4	1.8468	2.5485	3.2051	3.6097	3.9679	4.3926	4.6867	5.3052	7 3 3
5	1.0132	1.6716	2.2795	2.6511	2.9784	3.3649	3.6317	4.1909	2
6	.1829	.8691	1.4920	1.8689	2.1990	2.5870	2.8538	3.4109	1
7 5 5	1.7022	2.4277	3.1024	3.5164	3.8818	4.3140	4.6127	5.2394	7 2 2
6	.5970	1.3101	1.9574	2.3488	2.6914	3.0937	3.3703	3.9477	1
7 6 6	1.2568	2.0626	2.7972	3.2417	3.6303	4.0861	4.3989	5.0505	7 1 1
8 1 1	1.3418	2.1407	2.8691	3.3099	3.6953	4.1475	4.4579	5.1046	8 7 7
2	.7546	1.5117	2.1955	2.6060	2.9627	3.3785	3.6622	4.2489	6
3	.3518	1.0970	1.7686	2.1709	2.5201	2.9265	3.2034	3.7750	5
4	.0000	.7420	1.4099	1.8097	2.1564	2.5597	2.8343	3.4007	4
5		.3926	1.0619	1.4622	1.8093	2.2127	2.4872	3.0532	3
6		.0000	.6772	1.0818	1.4322	1.8391	2.1159	2.6860	2
7			.1516	.5703	.9319	1.3509	1.6353	2.2197	1

8 2 2	1•8059	2•5215	3•1876	3•5968	3•9581	4•3858	4•6815	5•3023	8 6 6
3	1•1888	1•8560	2•4716	2•8472	3•1774	3•5663	3•8340	4•3930	5
4	•7508	1•4053	2•0080	2•3751	2•6976	3•0770	3•3380	3•8822	4
5	•3495	1•0031	1•6036	1•9691	2•2898	2•6669	2•9262	3•4664	3
6		•5758	1•1831	1•5518	1•8749	2•2543	2•5149	3•0574	2
7		•0000	•6345	1•0169	1•3505	1•7410	2•0085	2•5636	1
8 3 3	1•9792	2•6666	3•3113	3•7093	4•0620	4•4807	4•7710	5•3820	8 5 5
4	1•3205	1•9568	2•5481	2•9108	3•2310	3•6096	3•8712	4•4197	4
5	•8286	1•4543	2•0338	2•3887	2•7017	3•0714	3•3267	3•8612	3
6	•3377	•9700	1•5528	1•9087	2•2221	2•5917	2•8467	3•3799	2
7		•3545	•9639	1•3328	1•6559	2•0355	2•2965	2•8405	1
8 4 4	2•0276	2•7074	3•3462	3•7412	4•0916	4•5078	4•7965	5•4049	8 4 4
5	1•3058	1•9348	2•5197	2•8789	3•1963	3•5720	3•8320	4•3779	3
6	•7173	1•3432	1•9215	2•2755	2•5876	2•9565	3•2115	3•7459	2
7	•0000	•6628	1•2651	1•6301	1•9502	2•3268	2•5862	3•1281	1
8 5 5	1•9792	2•6666	3•3113	3•7093	4•0620	4•4807	4•7710	5•3821	8 3 3
6	1•1462	1•7890	2•3836	2•7477	3•0688	3•4485	3•7110	4•2620	2
7	•3166	•9877	1•5971	1•9663	2•2900	2•6709	2•9332	3•4817	1
8 6 6	1•8059	2•5215	3•1876	3•5968	3•9581	4•3858	4•6815	5•3023	8 2 2
7	•7015	1•4041	2•0421	2•4280	2•7659	3•1631	3•4363	4•0071	1
8 7 7	1•3418	2•1407	2•8691	3•3099	3•6953	4•1475	4•4579	5•1047	8 1 1
9 1 1	1•4133	2•2067	2•9301	3•3679	3•7507	4•1999	4•5083	5•1511	9 8 8
2	•8494	1•6013	2•2805	2•6882	3•0426	3•4556	3•7375	4•3203	7
3	•4717	1•2112	1•8778	2•2771	2•6238	3•0272	3•3022	3•8698	6
4	•1522	•8872	1•5490	1•9452	2•2890	2•6888	2•9611	3•5229	5
5		•5825	1•2436	1•6392	1•9822	2•3811	2•6526	3•2126	4
6		•2668	•9307	1•3277	1•6718	2•0718	2•3440	2•9051	3
7			•5724	•9740	1•3219	1•7259	2•0008	2•5667	2
8			•0699	•4857	•8448	1•2610	1•5435	2•1239	1
9 2 2	1•8906	2•5984	3•2579	3•6633	4•0214	4•4454	4•7388	5•3549	9 7 7
3	1•3039	1•9634	2•5725	2•9444	3•2715	3•6570	3•9224	4•4768	6
4	•9012	1•5467	2•1419	2•5049	2•8239	3•1995	3•4579	3•9970	5
5	•5496	1•1912	1•7821	2•1422	2•4586	2•8308	3•0869	3•6208	4
6	•1983	•8424	1•4346	1•7952	2•1116	2•4838	2•7397	3•2730	3
7		•4521	1•0524	1•4171	1•7367	2•1123	2•3702	2•9072	2
8			•5348	•9133	1•2438	1•6307	1•8957	2•4458	1
9 3 3	2•0828	2•7596	3•3955	3•7885	4•1371	4•5513	4•8386	5•4440	9 6 6
4	1•4661	2•0912	2•6733	3•0310	3•3471	3•7211	3•9796	4•5221	5
5	1•0283	1•6397	2•2079	2•5567	2•8646	3•2289	3•4805	4•0080	4
6	•6272	1•2375	1•8033	2•1502	2•4563	2•8181	3•0680	3•5913	3
7	•1908	•8113	1•3838	1•7339	2•0422	2•4062	2•6573	3•1827	2
8		•2391	•8393	1•2029	1•5216	1•8963	2•1540	2•6915	1

9	4	4	2.1598	2.8249	3.4515	3.8398	4.1847	4.5949	4.8798	5.4809	9	5	5
		5	1.5015	2.1134	2.6845	3.0363	3.3476	3.7168	3.9724	4.5100			4
		6	1.0099	1.6105	2.1693	2.5129	2.8167	3.1768	3.4259	3.9494			3
		7	.5194	1.1267	1.6887	2.0332	2.3373	2.6971	2.9458	3.4678			2
		8			.5134	1.1023	1.4597	1.7734	2.1429	2.3975	2.9300			1
9	5	5	2.1598	2.8249	3.4515	3.8398	4.1847	4.5949	4.8798	5.4809	9	4	4
		6	1.4384	2.0512	2.6224	2.9740	3.2852	3.6541	3.9098	4.4476			3
		7	.8503	1.4597	2.0240	2.3701	2.6757	3.0377	3.2881	3.8144			2
		8	.1338	.7807	1.3691	1.7262	2.0397	2.4091	2.6639	3.1973			1
9	6	6	2.0828	2.7596	3.3955	3.7885	4.1371	4.5513	4.8386	5.4440	9	3	3
		7	1.2502	1.8816	2.4662	2.8246	3.1410	3.5155	3.7747	4.3193			2
		8	.4212	1.0810	1.6805	2.0439	2.3626	2.7381	2.9969	3.5388			1
9	7	7	1.8906	2.5984	3.2579	3.6633	4.0214	4.4454	4.7388	5.3550	9	2	2
		8	.7867	1.4813	2.1120	2.4936	2.8278	3.2207	3.4912	4.0568			1
9	8	8	1.4133	2.2067	2.9301	3.3679	3.7507	4.1999	4.5083	5.1511	9	1	1
10	1	1	1.4748	2.2637	2.9829	3.4182	3.7989	4.2456	4.5523	5.1916	10	9	9
		2	.9294	1.6772	2.3527	2.7581	3.1106	3.5214	3.8018	4.3816			8
		3	.5708	1.3059	1.9685	2.3656	2.7103	3.1115	3.3849	3.9493			7
		4	.2741	1.0040	1.6613	2.0549	2.3965	2.7937	3.0643	3.6226			6
		5	.0000	.7283	1.3839	1.7762	2.1165	2.5123	2.7818	3.3376			5
		6		.4554	1.1117	1.5043	1.8447	2.2405	2.5099	3.0654			4
		7		.1632	.8230	1.2176	1.5595	1.9569	2.2274	2.7848			3
		8			.4844	.8838	1.2297	1.6314	1.9046	2.4673			2
		9			.0000	.4134	.7706	1.1845	1.4654	2.0426			1
10	2	2	1.9618	2.6634	3.3176	3.7198	4.0753	4.4964	4.7877	5.4000	10	8	8
		3	1.3984	2.0519	2.6560	3.0251	3.3498	3.7325	3.9962	4.5470			7
		4	1.0207	1.6597	2.2495	2.6094	2.9259	3.2986	3.5551	4.0904			6
		5	.7013	1.3349	1.9192	2.2758	2.5891	2.9580	3.2118	3.7412			5
		6	.3971	1.0303	1.6136	1.9693	2.2818	2.6496	2.9026	3.4302			4
		7	.0781	.7154	1.3016	1.6587	1.9722	2.3410	2.5945	3.1230			3
		8		.3510	.9461	1.3077	1.6248	1.9974	2.2534	2.7863			2
		9			.4516	.8272	1.1551	1.5391	1.8022	2.3486			1
10	3	3	2.1673	2.8360	3.4649	3.8541	4.1995	4.6100	4.8950	5.4957	10	7	7
		4	1.5810	2.1979	2.7733	3.1272	3.4402	3.8107	4.0670	4.6050			6
		5	1.1783	1.7802	2.3407	2.6853	2.9897	3.3502	3.5993	4.1218			5
		6	.8269	1.4244	1.9803	2.3218	2.6235	2.9804	3.2272	3.7443			4
		7	.4759	1.0760	1.6330	1.9748	2.2766	2.6334	2.8799	3.3964			3
		8	.0749	.6869	1.2522	1.5980	1.9029	2.2629	2.5114	3.0313			2
		9		.1455	.7387	1.0984	1.4138	1.7848	2.0401	2.5727			1

TABLE I (continued)

10 4 4	2.2632	2.9174	3.5351	3.9183	4.2591	4.6647	4.9467	5.5421	10 6 6
5	1.6469	2.2472	2.8089	3.1556	3.4627	3.8273	4.0799	4.6115	5
6	1.2092	1.7951	2.3423	2.6797	2.9785	3.3330	3.5787	4.0951	4
7	.8084	1.3929	1.9376	2.2729	2.5697	2.9217	3.1655	3.6778	3
8	.3724	.9674	1.5188	1.8572	2.1562	2.5103	2.7553	3.2695	2
9		.3976	.9769	1.3289	1.6381	2.0027	2.2540	2.7800	1
10 5 5	2.2920	2.9419	3.5562	3.9378	4.2771	4.6814	4.9624	5.5562	10 5 5
6	1.6340	2.2292	2.7864	3.1305	3.4355	3.7979	4.0492	4.5788	4
7	1.1425	1.7261	2.2705	2.6062	2.9035	3.2565	3.5013	4.0166	3
8	.6523	1.2426	1.7902	2.1266	2.4241	2.7768	3.0211	3.5348	2
9	.0000	.6309	1.2056	1.5549	1.8619	2.2242	2.4742	2.9982	1
10 6 6	2.2632	2.9174	3.5351	3.9183	4.2591	4.6648	4.9467	5.5421	10 4 4
7	1.5421	2.1430	2.7040	3.0498	3.3562	3.7200	3.9723	4.5039	3
8	.9543	1.5516	2.1054	2.4455	2.7462	3.1027	3.3498	3.8696	2
9	.2383	.8737	1.4517	1.8027	2.1112	2.4751	2.7263	3.2531	1
10 7 7	2.1673	2.8360	3.4649	3.8541	4.1995	4.6100	4.8950	5.4958	10 3 3
8	1.3350	1.9575	2.5345	2.8884	3.2010	3.5713	3.8278	4.3672	2
9	.5065	1.1577	1.7493	2.1081	2.4229	2.7940	3.0500	3.5865	1
10 8 8	1.9618	2.6634	3.3176	3.7198	4.0753	4.4964	4.7878	5.4001	10 2 2
9	.8584	1.5466	2.1713	2.5493	2.8805	3.2700	3.5383	4.0993	1
10 9 9	1.4748	2.2637	2.9829	3.4182	3.7989	4.2456	4.5523	5.1917	10 1 1